A SHORT NOTE ON HOPF ALGEBROIDS: EXT AND PRIMITIVES IN (TENSOR POWERS OF) THE AUGMENTATION IDEAL

Abstract. This is a short note on how to identify elements of $\mathrm{Ext}^n(A, A)$ as primitive element in a quotient of the n-th tensor power of the augmentation ideal I a Hopf algebroid (A, Γ). One can thus understand Adams's $d-$ and $e-$ as well as Laures's $f-$invariant this way. This concept is general, hence applicable to hypothetical follow-up invariants.

Let (A, Γ) be a Hopf algebroid, such that Γ is flat over A and let I denote the augmentation ideal, i.e. the kernel of the augmentation

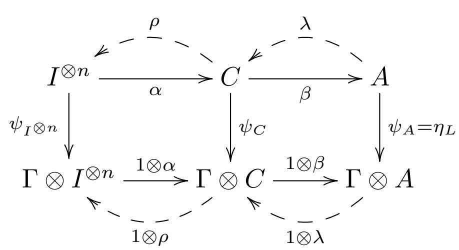
$$\epsilon\colon \Gamma \longrightarrow A.$$

Let (A, Γ) be connected, i.e ϵ is an isomorphism in degree less than or equal to zero.

Theorem 1. *We can realize the isomorphism*

$$\omega\colon \mathrm{Ext}_\Gamma^{n+1,q}(A, A) \longrightarrow P(I^{\otimes n+1})/\sim$$

as follows: For $S \in \mathrm{Ext}_\Gamma^{n+1,q}(A, A)$ we construct $\omega_S := \omega(S)$ as a defect class of the commutativity of the A-module splitting (ρ, λ) and the Γ-comodule action ψ_C in



Proof. I is a free A-module on (a priori) infinitely many generators $(g_i)_{i\in\Im}$. (This means the cardinality of $\Im$ may be infinite.)
A generator in $I^{\otimes n} = I_1 \otimes \cdots \otimes I_n$, where I_k just indicates the k-th copy of I in the tensor product, is therefore a combination $g_{i_1} \otimes \cdots \otimes g_{i_n}$, for $g_{i_k} \in I_k$.
Be aware that, since the Hopf algebroid (A, Γ) is connected, the degree of such a g_{i_k} is positive.
Using Lemma 4 we can work with a short exact sequence S_1 of the form

$$S_1\colon \quad I^{\otimes n} \xrightarrow{\alpha} C \xrightarrow{\beta} A, \tag{1}$$

which is the image of our $(n + 1)$-extension $\in \mathrm{Ext}_\Gamma^{n+1,q}(A, A)$ under the sequence of isomorphisms described in Lemma 4.

Set $r = q - n$. The second index r in $\mathrm{Ext}_\Gamma^{1,r}(A, I^{\otimes n})$ requires

$$deg(\alpha) + deg(\beta) = -r. \tag{2}$$

So if $deg(\alpha) = p$, it follows that $deg(\beta) = -(p+r)$. Remark that under each sequence of Γ-comodules there is an underlying sequence of A-modules, which in the case of S_1 splits, since A is a free module of rank one over itself (and free implies projective). We choose a splitting and call the splitting maps ρ and λ.
So C has an A-module representation

$$C \cong I^{\otimes n} \cdot (x_i)_{i \in \Im} \oplus A \cdot y \tag{3}$$

where $x_i = \alpha(h_i)$ for generators $h_i = g_{i_1} \otimes \cdots \otimes g_{i_n}$ of $I^{\otimes n}$ and $y = \lambda(1)$ for the generator 1 of A.
Considering S_1 and its splitting together with the natural Γ-coactions of the objects in S_1 and the flatness of Γ over A, we obtain the diagram in the theorem.
The Γ-coaction of the generator $y \in C$ is generally given by

$$\psi_C(y) = 1 \otimes y + \sum_{|y'|+|y''|=|y|} y' \otimes y'',$$

where $y' \in I \subset \Gamma$ and $y'' \in C$, such that $0 < |y'|, |y''| < |y|$.
By the commutativity of the right square in the diagram we see that

$$(1 \otimes \beta)(\psi_C(y)) = 1 \otimes 1$$

Therefore by exactness of the bottom short exact sequence we get

$$(1 \otimes \rho)(\psi_C(y)) = \sum_{|y'|+|z''|=r} y' \otimes z''$$

If we substitute $z'' \in I^{\otimes n}$ by its representation $\sum a_j h_j$ in terms of generators h_j of $I^{\otimes n}$ of suitable degrees, then

$$\begin{aligned}(1 \otimes \rho)(\psi_C(y)) &= \sum y' \otimes \sum a_j h_j \\ &= \sum \sum y' \cdot a_j \otimes h_j, \\ &= \sum w_j \otimes h_j\end{aligned}$$

for $w_j = \sum y_j \cdot a_j$. Thus we obtain for the Γ-coaction of y

$$\psi_C(y) = 1 \otimes y + \sum_{|w_j|+|x_j|=|y|} w_j \otimes x_j. \tag{4}$$

Remark that $w_S = (1 \otimes \rho)(\psi_C(y))$ is an element of $I^{\otimes n+1} = I \otimes I^{\otimes n} \subset \Gamma \otimes I^{\otimes n}$, since $y' \in I$ and $h_j \in I^{\otimes n}$. Since the Γ-coaction of $I^{\otimes n+1}$ is given by

$$\psi_{I^{\otimes n+1}} = \psi_\Gamma \otimes 1 - 1 \otimes \psi_{I^{\otimes n}}$$

we calculate

$$\begin{aligned}\psi_{I^{\otimes n+1}}(w_S) &= \psi_\Gamma \otimes 1(w_S) - 1 \otimes \psi_{I^{\otimes n}}(w_S) \\ &= 1 \otimes \Big(\sum w_j \otimes h_j\Big) + \sum \Big(\sum w_j' \otimes w_j''\Big) \otimes h_j - \sum w_j \otimes \Big(\sum h_j' \otimes h_j''\Big).\end{aligned}$$

By Lemma 3 the two latter terms are equal, and hence $w_S \in I^{\otimes n+1}$ is primitive.

So far this depends on the choice of splitting. Proposition **??** takes care of this by specifying the necessary equivalence relation that one has to divide out.

Conversely we see very easily that each primitive element w (representing its residue class) induces a comodule structure on C by

$$\psi_C(x_i) = (1 \otimes \alpha)(\psi_{I^{\otimes n}}(h_i))$$
$$\psi_C(y) = 1 \otimes y + (1 \otimes \alpha)(w).$$

□

Lemma 2. *The Γ-coactions of $h_i \in I^{\otimes n}$ and $x_i \in C$ are given by*

$$\psi_{I^{\otimes n}}(h_i) = 1 \otimes h_i + \sum_{\substack{i,j \\ |\bar{h}_{i,j}|+|h_j|=k}} \bar{h}_{i,j} \otimes h_j$$

$$\psi_C(x_i) = 1 \otimes x_i + \sum_{\substack{i,j \\ |\bar{h}_{i,j}|+|x_j|=|x_i|}} \bar{h}_{i,j} \otimes x_j. \tag{5}$$

Proof. Suppose the degree of the generator $h_i \in I^{\otimes n}$ is $k = |g_{i_1}| + \cdots + |g_{i_n}|$. Then the general Γ-coaction of h_i looks like

$$\psi_{I^{\otimes n}}(h_i) = 1 \otimes h_i + \sum_{\substack{h'_i, h''_i \\ |h'_i|+|h''_i|=k}} h'_i \otimes h''_i,$$

where $h'_i \in I$, $h''_i \in I^{\otimes n}$ and $0 < |h'_i|, |h''_i| < k$. Further all h''_i are of the following form

$$h''_i = \sum_{\substack{j \\ |h'_i|+|h_j|=k}} a_j h_j, \tag{6}$$

which implies

$$\psi_{I^{\otimes n}}(h_i) = 1 \otimes h_i + \sum_{\substack{h'_i, h''_i \\ |h'_i|+|h''_i|=k}} h'_i \otimes \sum_{\substack{j \\ |h'_i|+|h_j|=k}} a_j h_j$$
$$= 1 \otimes h_i + \sum_{\substack{h'_i, h''_i \\ |h'_i|+|h''_i|=k}} \sum_{\substack{j \\ |h'_i|+|h_j|=k}} h'_i \cdot a_j \otimes h_j.$$

Set

$$\bar{h}_{i,j} = \sum_{\substack{j \\ |h'_i|+|h_j|=k}} h'_i \cdot a_j \tag{7}$$

to obtain

$$\psi_{I^{\otimes n}}(h_i) = 1 \otimes h_i + \sum_{\substack{i,j \\ |\bar{h}_{i,j}|+|h_j|=k}} \bar{h}_{i,j} \otimes h_j.$$

By the commutativity of the left square of the diagram we now see that the Γ-coaction of the generators $x_i \in C$ is given by

$$\psi_C(x_i) = 1 \otimes x_i + \sum_{\substack{i,j \\ |\bar{h}_{i,j}|+|x_j|=|x_i|}} \bar{h}_{i,j} \otimes x_j.$$

□

Lemma 3.

$$\sum \left(\sum w'_j \otimes w''_j\right) \otimes h_j = \sum w_j \otimes \left(\sum h'_j \otimes h''_j\right)$$

Proof. We use the coassociativity condition of ψ_C applied to the generator y

$$(\psi_\Gamma \otimes 1) \circ \psi_C(y) = (1 \otimes \psi_C) \circ \psi_C(y). \tag{8}$$

Using (4) we are able to calculate both sides as

$$(\psi_\Gamma \otimes 1) \circ \psi_C(y) = 1 \otimes 1 \otimes y + 1 \otimes \left(\sum w_j \otimes x_j\right) + \sum w_j \otimes 1 \otimes x_j + \sum \sum w'_j \otimes w''_j \otimes x_j$$

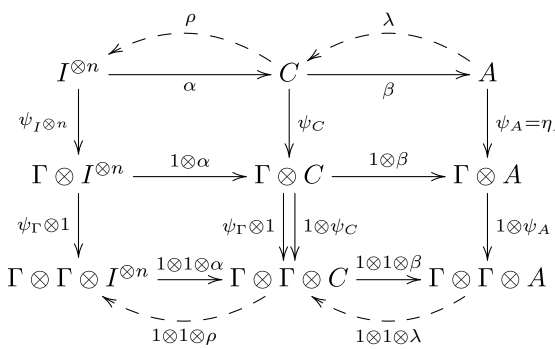
and further using (5) as

$$(1 \otimes \psi_C) \circ \psi_C(y) = 1 \otimes 1 \otimes y + 1 \otimes \left(\sum w_j \otimes x_j\right) + \sum w_j \otimes (1 \otimes x_j + \sum \bar{h}_{i,j} \otimes x_i).$$

From equality it follows now that

$$\sum \left(\sum w'_j \otimes w''_j\right) \otimes x_j = \sum w_j \otimes (\sum \bar{h}_{i,j} \otimes x_i).$$

From



we see that applying $(1 \otimes 1 \otimes \rho)$ to both sides of (8) we get

$$\sum \left(\sum w'_j \otimes w''_j\right) \otimes h_j = \sum w_j \otimes (\sum \bar{h}_{i,j} \otimes h_i).$$

And using (6) and (7) we see that the latter is

$$\sum w_j \otimes (\sum \bar{h}_{i,j} \otimes h_i) = \sum w_j \otimes (\sum h'_j \otimes h''_j).$$

Thus we obtain

$$\sum \left(\sum w'_j \otimes w''_j\right) \otimes h_j = \sum w_j \otimes (\sum h'_j \otimes h''_j).$$

□

Lemma 4.

$$\mathrm{Ext}_\Gamma^{n+1,q}(A, A) \cong \mathrm{Ext}_\Gamma^{n,q-1}(A, I) \cong \cdots \cong \mathrm{Ext}_\Gamma^{1,q-n}(A, I^{\otimes n})$$

Proof. We consider the short exact sequence

$$I^{\otimes n+1} \longrightarrow I^{\otimes n} \longrightarrow \Gamma \otimes I^{\otimes n}.$$

Applying the left exact functor $\mathrm{Hom}_\Gamma(\mathrm{A}, -)$ yields a long exact sequence, where $\mathrm{Ext}_\Gamma^{n+1}(A, \Gamma \otimes I^{\otimes n}) \cong 0$ for $n \geq 0$, since $\Gamma \otimes I^{\otimes n}$ is a relatively injective comodule, even an extended one (cf Lemma A1.2.8 (b) in Ravenel '86). □

Proposition 5. *The equivalence relation* $\sim$ *on the primitive elements in* $I^{\otimes n+1}$ *of Theorem 1 is given as follows:*

$$w = \sum w_j \otimes h_j \sim \sum \bar{w}_j \otimes h_j = \bar{w}$$

if and only if for all j *there exist* $a_j \in A$*, such that*

$$\sum \bar{w}_j \otimes h_j = \sum w_j \otimes h_j + \sum \eta_L(a_j)\psi_{I^{\otimes n}}(h_j) - \sum \eta_R(a_j) \otimes h_j.$$

Proof. When we say that the short exact sequence (1) splits and use the A-module representation (3) of the Γ-comodule C, we actually make a choice (ρ, λ) of such a splitting. We could obviously choose another splitting $(\bar{\rho}, \bar{\lambda})$, i.e. $\bar{\lambda}(1) = \bar{y} = y + \sum a_j x_j$, and since the Γ-comodule structure of C must not depend on the chosen A-module splitting, the generators y and $\bar{y}$ should induce the same primitive element $w = \bar{w}$.
In general this will not be the case and hence we have to introduce an equivalence relation on the group $P(I^{\otimes n+1})$ of primitive elements, such that w and $\bar{w}$ are in the same residue class. This equivalence relation can be derived as follows.
With the same argumentation we used for y we see that $\bar{y}$ gives

$$\psi_C(\bar{y}) = 1 \otimes \bar{y} + \sum \bar{w}_j \otimes x_j$$

for the coaction of $\bar{y}$. On the other hand we can use the representation $\bar{y} = y + \sum a_j x_j$ to compute

$$\begin{aligned}
\psi_C(\bar{y}) &= \psi_C(y + \sum a_j \otimes x_j) \\
&= \psi_C(y) + \sum a_j \psi_C(x_j) \\
&= 1 \otimes y + \sum w_j \otimes x_j + \sum \eta_L(a_j)\psi_C(x_j) + \sum 1 \otimes \eta_L(a_j)x_j - \sum 1 \otimes \eta_L(a_j)x_j \\
&= 1 \otimes \bar{y} + \sum w_j \otimes x_j + \sum \eta_L(a_j)\psi_C(x_j) - \sum \eta_R(a_j) \otimes x_j.
\end{aligned}$$

When we now apply $(1 \otimes \rho)$ to $\psi_C(y)$ we get the following equation.

$$\sum \bar{w}_j \otimes h_j = \sum w_j \otimes h_j + \sum \eta_L(a_j)\psi_{I^{\otimes n}}(h_j) - \sum \eta_R(a_j) \otimes h_j.$$

□

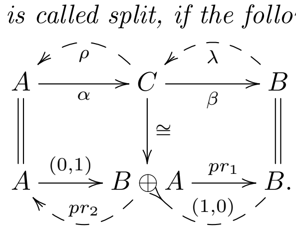
Remark 1. *For* $n = 0$ *this specializes to the well known result*

$$\mathrm{Ext}^{1,q}_{\Gamma}(A, A) \xrightarrow{\cong} P(E_*\bar{E})_q/(\eta_L - \eta_R)(A)_q,$$

since there is only one h_j*, the generator* $1 \in A$*, satisfying*

$$\psi_{I^{\otimes 0}}(1) = \psi_A(1) = \eta_L(1) = 1 \otimes 1.$$

Remark 2. *This is a short reminder about splitt short exact sequences.*
A short exact sequence is called split, if the following diagram commutes



This definition has the advantage that it is clear that the splitting maps (ρ, λ) *form a short exact sequence by theirselves, since the other splitting is exact and the diagram*

is commutative. In particular we can construct for a given λ *with* $\beta \circ \lambda = 1$ *the corresponding map* ρ *and vice versa.*
Splittings are not unique, we always have to choose a λ*, i.e. image* y *of* λ*.*